- Quick Recap.
  - Importance of Deductive and Inductive Logic
  - Bayes' theorem:Simple applications
  - Parameter Determination and Hypothesis Testing
  - Some useful distributions: Likelihood Functions used in particle physics
- Today
  - What are 'good-estimates' for a given distribution
  - Parameter Determination and Hypothesis Testing
  - Straight Line Fit and Outliers
  - Error Determination, and Propagation
  - Correlated Variables and Errors, error matrix

- How do we know what estimate is the best estimate?
  - Assume probability is maximum for the best estimate
  - Probability of points in nbd. obtained by making a Taylor expansion about the max. probability

- P=P(X|{data},I), then best estimate of its value X_0 is obtained by maximising L=ln[P(X|{data};I)]

$$P(X\,|\,\{data\},I) \approx A\exp\left[\frac{1}{2}\frac{d^2L}{dX^2}\Big|_{X_0}(X-X_0)^2\right]$$

$$\sigma = \left(-\frac{d^2L}{dX^2}\right)_{X_0}^{-1/2};$$

X = X_0 ±σ, best estimate is X_0 and σ is error (~68% chance that true value within this)

What if the distribution is asymmetric or multimodal ?

- Apply this to the experiment : flipping the coin

$$P(H\,|\,\{data\},I) \propto \underbrace{H^{R}(1-H)^{N-R}}_{Likelihood\ function}$$

$$\frac{dL}{dH}\Big|_{H_0} = \frac{R}{H_0} - \frac{(N-R)}{1-H_0} = 0 \Rightarrow H_0 = \frac{R}{N}$$

$$\frac{d^2L}{dH^2}\Big|_{H_0} = -\frac{R}{H_0^{\,2}} - \frac{(N-R)}{(1-H_0)^2} = \frac{-N}{H_0(1-H_0)} \Rightarrow \sigma = \sqrt{\frac{H_0(1-H_0)}{N}}$$

$$\therefore width \propto \frac{1}{\sqrt{N}}$$

Numerator maximum at $H_0 = 0.5$

- Assume data distributed according to a Gaussian
  - Calculate the mean and the error
  - Common sense 'mean' $= \frac{1}{N}\sum_{k=1}^{N} x_k$

$$\left.\frac{dL}{d\mu}\right|_{\mu_0} = 0 \Rightarrow \sum_{k=1}^{N} x_k = N\mu_0 \Rightarrow \mu_0 = \frac{1}{N}\sum_{k=1}^{N} x_k$$

$$\left.\frac{d^2L}{d\mu^2}\right|_{\mu_0} = -\sum_{k=1}^{N}\frac{1}{\sigma^2} = -\frac{N}{\sigma^2}$$

$$\therefore \mu = \mu_0 \pm \frac{\sigma}{\sqrt{N}}$$

- But we do not know σ; two unknowns

$$\mu_0 = \frac{1}{N}\sum_{k=1}^{N} x_k$$

$$\mu = \mu_0 \pm \frac{S}{\sqrt{N}} \quad where\ S^2 = \frac{1}{N-1}\sum_{k=1}^{N}(x_k - \mu)^2$$

- Binned data:

$$\mu_0 = \frac{\sum_k n_k x_k}{\sum_k n_k}$$

$$and\ S^2 = \frac{\sum_k n_k (x_k - \mu)^2}{\sum_k n_k - 1}, \quad \mu = \mu_0 \pm \sqrt{\frac{S^2}{\sum_k n_k}}$$

For continuous distribution

$$\mu_0 = \frac{\int n(x)\ x\ dx}{N} \qquad and\ S^2 = \frac{1}{N}\int n(x)(x-\mu)^2 dx$$

$$and\ N = \int n(x) dx$$

- What if errors on each x_k are all different
- Again use maximum likelihood

- And if individual errors are different then

$$P(x_k \mid \mu, \sigma_k) = \frac{1}{\sigma_k \sqrt{2\pi}} \exp\left[-\frac{(x_k - \mu)^2}{2\sigma_k^{\,2}} \right]$$

$$\& \quad \mu_0 = \frac{\sum_{k=1}^{N} w_k x_k}{\sum_{k=1}^{N} w_k} \qquad where\ w_k = \frac{1}{\sigma_k^2}$$

$$\& \quad \mu = \mu_0 \pm \left(\sum_{k=1}^{N} w_k \right)^{-1/2}$$

Caution:
Measured counting rate
1 ± 1 in 1st hour and
100 ± 10 in 2nd hour
Average counting rate?

- Other methods, examples
  - Moments

$$\frac{dn}{d\cos\theta} = a + b\cos^2\theta$$

$$\frac{b}{a} = \frac{5(3\overline{\cos^2\theta} - 1)}{3 - 5\overline{\cos^2\theta}}$$

$$\delta = \frac{1}{\sqrt{n}}\sqrt{\left[\frac{1}{n-1}\sum_{k=1}^{n}(\cos^2\theta_k - \overline{\cos^2\theta_k})\right]}$$

Hypothesis

  - Likelihood (normalization constant)

$$L\left(\frac{b}{a}\right) = \prod_{k=1}^{n} y_k \quad ; \qquad y_k = N\left(1 + \left(\frac{b}{a}\right)\cos^2\theta\right)$$

- Least Squares Method
- Assume
  - Each data point is independent
  - Noise associated with experimental measurement is Gaussian

$$P(D_k \mid X, I) = \frac{1}{\sigma_k \sqrt{2\pi}} \exp\left[-\frac{(F_k - D_k)^2}{2\sigma_k^2}\right]$$

$$F_k = f(X, k) \quad e.g. \quad f(X, k) = y = mx_k + c$$

$$P(D_k \mid X, I) \propto \exp(-\frac{\chi^2}{2})$$

$$\chi^2 = \sum_{k=1}^{N} \left(\frac{F_k - D_k}{\sigma^k}\right)^2$$

Obtain set of values of X, the parameters, by minimising. Useful for fitting distribution

- Straight Line Fit

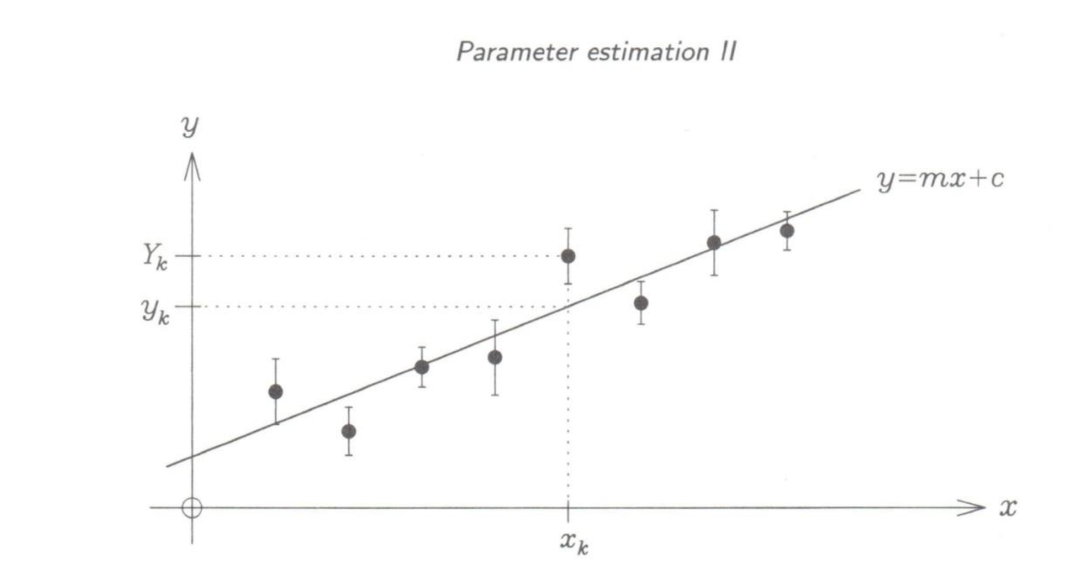

Fig. 3.12 Fitting a straight line to noisy graphical data.

$$\chi^2 = \sum_{k=1}^{N} \frac{(mx_k + c - Y_k)^2}{\sigma_k^2}$$

Minimise to get values of m and c

- If Y_k Poisson distributed, then $\sigma_k^2 = Y_k$

(σ should be error on theoretical estimate or on measured value?)

What if there are too many outliers?

Least-squares extensions

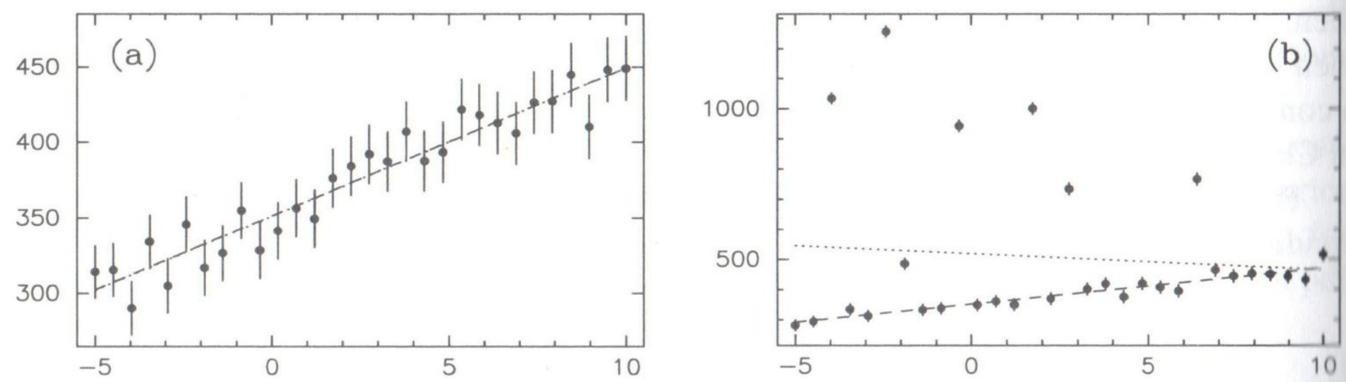

Fig. 8.1 The problem of outliers: (a) a 'well-behaved' set of data; (b) a case where quirky things occasionally happen. The least-squares estimate of the best straight lines is indicated by the dots, whereas the corresponding results following the analysis in Section 8.3.1 is marked by the dashes.

$$L = \log_e[P(X \mid D, I)] = c + \sum_{k=1}^{N} \log_e\left[\frac{1 - e^{-R_k^2/2}}{R_k^2}\right]$$

$$R_k = \frac{(F_k - D_k)}{\sigma_0} \quad , \quad P(\sigma \mid \sigma_0, I) = \frac{\sigma_0}{\sigma^2} \quad \textit{for} \;\; \sigma \geq \sigma_0$$

Assumed a lower bound on σ

- What do errors tell us? Why estimate errors?
  - Usefulness of measurement (J/ψ mass=3.0969 GeV)
- Errors on parameters

lone measurement?

  - the mean charged particle multiplicity
  - Temperature
- Multiplicity distribution is assumed Gaussian
  - Peak is the most likely value N_m
  - 68.3 % probability that true value N_0 is in the range $N_m \pm \sigma$
  - 90% confidence level that $N_0 \leq N_m + 1.28\,\sigma$
  - 95% confidence level that $N_0 \leq N_m + 1.64\,\sigma$