Phonon contribution to the thermal conductivity in neutron star core

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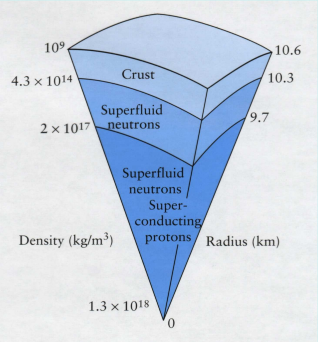
Plan of the discussion



Outline

- Motivation
- Superfluid phonon in the core of neutron star
- Thermal conductivity (κ)
  - Boltzmann equation
  - Superfluid phonon dispersion law: beyond leading order
  - Temperature dependence of κ
- Results
- Summary & Conclusions

Motivation

- Neutron stars are compact stars of mass $\sim 1.4\, M_\odot$ and radius ~ 10 km
- A neutron star is thought to consist of a thin crust ($\sim 1\%$ by mass) and a bulky core
- The core extends from the layer of the density $\rho \approx 0.5\, \rho_0$ to the stellar center $[\rho \sim 10\, \rho_0]$, where ρ_0 is the nuclear matter density.
- Neutron star cores $\Rightarrow$ neutrons, with an admixture of protons, electrons and muons. This makes neutron stars unique natural laboratories of dense matter.

continued $\cdots$

κ is important for cooling of isolated neutron stars

Cooling processes

Cooling of young neutron star

The cooling is realized via two channels $\Rightarrow$ by neutrino emission from the neutron star core and by transport of heat from the internal layers to the surface resulting in the thermal emission of photons.

- Powerful neutrino emission
- Thermal conduction

The equation controlling the time evolution of the neutron star temperature

$$C_v \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial r^2} + \frac{1}{r^2} \frac{\partial (r^2 \kappa)}{\partial r} \frac{\partial T}{\partial r} - Q_\nu$$

Superfluid phonon

- Migdals observation $\Rightarrow$ at low temperatures superfluidity of neutron matter may occur in the core of compact stars.
- Due to the onset of superfluidity$\Rightarrow$ a collective mode appears $\Rightarrow$ superfluid phonon.

Superfluid phonon $\Rightarrow$ Goldstone mode associated to the spontaneous symmetry breaking of a $U(1)$ symmetry, which corresponds to particle number conservation.

Effective Lagrangian for superfluid phonon $\Rightarrow$ expansion in derivatives of the Goldstone field

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\left((\partial_t\phi)^2 - v_{\rm ph}^2(\nabla\phi)^2\right) - g\left((\partial_t\phi)^3 - 3\eta_g\,\partial_t\phi(\nabla\phi)^2\right) \\ &+ \lambda\left((\partial_t\phi)^4 - \eta_{\lambda,1}(\partial_t\phi)^2(\nabla\phi)^2 + \eta_{\lambda,2}(\nabla\phi)^4\right) + \cdots\end{aligned}$$

- $\phi \Rightarrow$ scalar phonon field
- Phonon selfcouplings can be expressed in terms of the speed of sound at $T=0$ and derivatives with respect to density

Thermal Conductivity

- In a system $\Rightarrow$ temperature distribution is not uniform, Thermal Conductivity relates heat flow to the negative gradient in the temperature.
- In hydrodynamics heat flow $\Rightarrow$ $\mathbf{q} = -\kappa \nabla T$
- In kinetic theory heat flow $\Rightarrow$ $\mathbf{q} = \int \frac{d^3p}{(2\pi)^3} \mathbf{v}_p\, E_p\, \delta f_p$
  - Particle velocity $\mathbf{v}_p = \partial E_P / \partial \mathbf{p}$, $\delta f_p = f_p - f_p^0$
  - local thermal equilibrium $f_p^0 = 1/(e^{p_\mu u^\mu / T} - 1)$
- Non-equilibrium distribution function $\Rightarrow$ thermal gradient in the medium $\delta f_p = -\frac{f_p^0(1+f_p^0)}{T^3} g(p) p \cdot \nabla T$

$$\kappa = \frac{1}{3T^3} \int \frac{d^3p}{(2\pi)^3} f_p(1+f_p) g(p)\ \ v_p\, E_p\, p.$$

T. Schafer et. al Phys. Rev. C 81, 045205 (2010)

continued ⋯

- $g(p)$ is dimensionless variable.
- $g(p)$ obtained solving Boltzmann equation

  - Boltzmann equation

$$\frac{df_p}{dt} = \frac{\partial f_p}{\partial t} + \frac{\partial E_p}{\partial \mathbf{p}} \cdot \nabla f_p = C[f_p]$$

  - Collision integral

$$C[f_p] = \frac{1}{2E_p} \int_{p',k,k'} (2\pi)^4 \delta^{(4)}(P + K - P' - K') \frac{1}{2} |\mathcal{M}|^2 D$$

  - Phase-space factor

$$D = f_{p'} f_{k'} (1 + f_p)(1 + f_k) - f_p f_k (1 + f_{p'})(1 + f_{k'})$$

  - $\mathcal{M} \leftrightarrow$ scattering matrix

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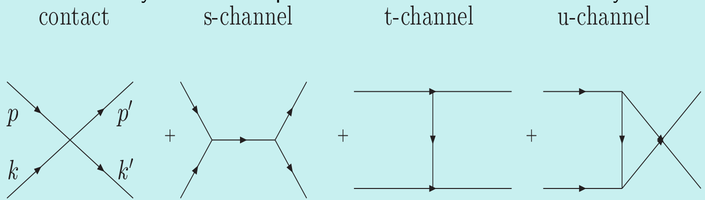
Superfluid phonon dispersion law: beyond leading order

Solution to the Boltzmann equation obeys the constraints of both energy and momentum conservation

$$\int \frac{d^3p}{(2\pi)^3} E_p\,\delta f_p = \int \frac{d^3p}{(2\pi)^3} p\,\delta f_p = \frac{\nabla T}{3\,T^3} \int \frac{d^3p}{(2\pi)^3}\, f_p(1+f_p)g(p)\,p^2 = 0$$

- Phonons $\Rightarrow$ exactly linear dispersion relation do not contribute to the thermal conductivity
- Beyond leading order $\Rightarrow$ $E_p = c_s p\left(1+\gamma p^2\right)$ where, $\gamma = -\frac{c_s^2}{15\Delta^2}$,
- Phonon dispersion law curves downward beyond linear order $\Rightarrow$ collisional processes of $1 \to 2$ kinematically forbidden
- Relevant binary collisions of phonons for the thermal conductivity are



Kinematically forbidden processes

<u>**Sign of γ plays a crucial role in determining which processes are allowed**</u>

For $1 \to 2$ processes energy and momentum conservation impose

$$\begin{aligned} E_a &= E_b + E_c \\ \vec{p}_a &= \vec{p}_b + \vec{p}_c \end{aligned}$$

- Beyond leading order $\Rightarrow E_p = c_s p \left(1 + \gamma p^2\right)$
- First order in $\gamma \Rightarrow$ the NLO correction $\Rightarrow$

$$\theta_{bc} = \sqrt{6\gamma}\,(p_b + p_c)$$

- For the one to two processes to be kinematically allowed, it is necessary that $\gamma > 0$.

Variational solution to the Boltzmann equation

- $g(p)$ in a basis of orthogonal polynomials $\Rightarrow g(p) = \sum_s b_s B_s(p^2)$
- The polynomials are orthogonal with regard to the inner product

$$\int d\Gamma f_p(1+f_p)p^2 B_s(p^2)B_t(p^2) \equiv A_s\delta_{st}$$

$$\kappa = \left(\frac{4a_1^2}{3T^2}\right) A_1^2 M_{11}^{-1}$$

- $M_{11}^{-1} \Rightarrow$ (1,1) element of inverse of the truncated $N \times N$ matrix. The bound is saturated as $N \to \infty$.

$$\begin{aligned} M_{st} &= \int d\Gamma_{p,k,k',p'} \mathbf{Q}_s \cdot \mathbf{Q}_t \\ \mathbf{Q}_s &= B_s(p^2)\mathbf{p} + B_s(k^2)\mathbf{k} - B_s(k'^2)\mathbf{k}' - B_s(p'^2)\mathbf{p}' \end{aligned}$$

- Phonons with non-linear dispersion relation

$$a_1 = \frac{4c_s^4}{15\Delta^2} \qquad A_1 = \frac{256\pi^6}{245c_s^9}T^9$$

Scattering matrices

Contact amplitude

$$
\begin{aligned}
i\mathcal{M}_{c.t.} &= -i\lambda\Big\{24E_pE_kE_{p'}E_{k'} - 4\eta_{\lambda,1}\left(E_pE_k\ \mathbf{p}'\cdot\mathbf{k}' + E_pE_{p'}\ \mathbf{k}\cdot\mathbf{k}'\right.\\
&+ \left. E_pE_{k'}\ \mathbf{p}'\cdot\mathbf{k} + E_{p'}E_k\ \mathbf{p}\cdot\mathbf{k}' + E_{p'}E_{k'}\ \mathbf{p}\cdot\mathbf{k} + E_kE_{k'}\mathbf{p}\cdot\mathbf{p}'\right)\\
&+ 8\eta_{\lambda,2}\left(\mathbf{p}\cdot\mathbf{k}\ \mathbf{p}'\cdot\mathbf{k}' + \mathbf{p}\cdot\mathbf{p}'\ \mathbf{k}\cdot\mathbf{k}' + \mathbf{p}\cdot\mathbf{k}'\ \mathbf{p}'\cdot\mathbf{k}\right)\Big\}
\end{aligned}
$$

s-channel amplitude

$$
\begin{aligned}
i\mathcal{M}_s &= -4ig^2G(P+K)\left\{E_pK^2 + E_kP^2 + 2(E_p+E_k)P\cdot K\right\}\\
&\left\{E_{p'}K'^2 + E_{k'}P'^2 + 2(E_{k'}+E_{p'})P'\cdot K'\right\}
\end{aligned}
$$

- $G \Rightarrow$ phonon propagator.
- The t- and u-channel amplitudes can be obtained from the s-channel one by using the crossing symmetry $i\mathcal{M}_t = i\mathcal{M}_s(K \leftrightarrow -P')$ and $i\mathcal{M}_u = i\mathcal{M}_s(K \leftrightarrow -K')$.

Phonon propagator

- Leading order phonon propagator

$$\mathcal{G}_{\rm ph}\left(p_i^0+p_j^0,\vec{p}_i+\vec{p}_j\right)=\frac{1}{(p_i^0+p_j^0)^2-E^2_{p_i+p_j}}$$

- Next to leading order phonon propagator

$$\begin{aligned}\mathcal{G}_{\rm ph}\left(p_i^0+p_j^0,\vec{p}_i+\vec{p}_j\right) &= i\Bigg[c_s^2(p_i+p_j)^2\left[1+2\gamma\left(\frac{p_i^3+p_j^3}{p_i+p_j}\right)\right] \\ &- c_s^2\left(\vec{p}_i+\vec{p}_j\right)^2\left[1+2\gamma\left(\vec{p}_i+\vec{p}_j\right)^2\right]\Bigg]^{-1}\end{aligned}$$

- In the collinear region $\Rightarrow \theta_{ij}\approx 0$ the propagator behaves as $\sim 1/p^4$.
- Region of large angle scattering the propagator behaves as $\sim 1/p^2$.

Temperature dependence of the thermal conductivity

$$\text{Temperature dependence} \Rightarrow |\mathcal{M}|^2 \propto T^{12} \times \frac{1}{G^2}$$

- For large angle collisions $\Rightarrow G^2 \propto T^{-4} \rightarrow |\mathcal{M}|^2 \propto T^8$,

$$\kappa \propto \frac{T^{16}}{\Delta^4}\frac{1}{T^{18}} \propto \frac{1}{T^2\Delta^4} \qquad \text{for large angle collisions.}$$

- In collinear region $\Rightarrow G^2 \propto \Delta^4 T^{-8} \rightarrow |\mathcal{M}|^2 \propto T^4\Delta^4$,

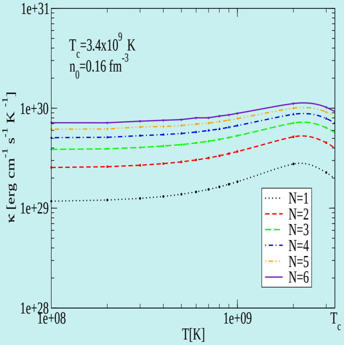
$$\kappa \propto \frac{T^{16}}{\Delta^4}\frac{1}{T^{14}\Delta^4} \propto \frac{T^2}{\Delta^8} \qquad \text{for small angle collisions.}$$

- In combined large-small angle collisions $\Rightarrow G^2 \propto \Delta^2 T^{-6} \rightarrow |\mathcal{M}|^2 \propto T^6\Delta^2$,

$$\kappa \propto \frac{T^{16}}{\Delta^4}\frac{1}{T^{16}\Delta^2} \propto \frac{1}{\Delta^6} \qquad \text{for combined large} - \text{small angle collisions}$$

Results

Variational solution of κ



- Speed of sound at $T = 0$ and the different phonon selfcouplings $\Rightarrow$ the EoS for neutron matter in neutron stars.
- Nucleonic EoS $\Rightarrow$ APR98 Akmal, Pandharipande and Ravenhall Phys. Rev. C 58, 1804 (1998)
- The final value of the number $N \Rightarrow$ imposed the deviation with respect to the previous order should be $\lesssim 10\%$.
- For $T \lesssim 10^9 K$, below T_c, $\kappa \Rightarrow$ almost independent of T, with subleading corrections $\sim T$ and T^2.

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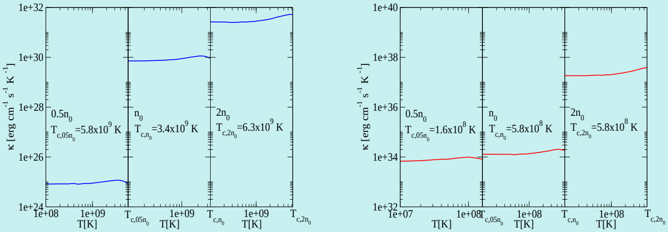
Phonon contribution to κ



- For $N = 6 \Rightarrow$ a fit to our numerical results $\Rightarrow$
  $\kappa \sim (7.02 \times 10^{29} + 9.28 \times 10^{19}\ T + 9.08 \times 10^{10}\ T^2)\ \mathrm{erg\ cm^{-1}\ s^{-1}\ K^{-1}}$.
- For both the gaps $\Rightarrow$ the dominant processes to the phonon contribution to the thermal conductivity corresponds to the combined small and large angle collisions $\Rightarrow$ T independent behaviour of κ.
- The thermal conductivity grows with increasing density, with a non-linear dependence.

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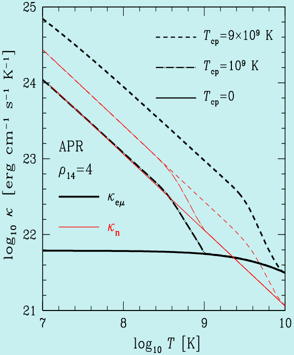
Electromagnetic contributions to κ



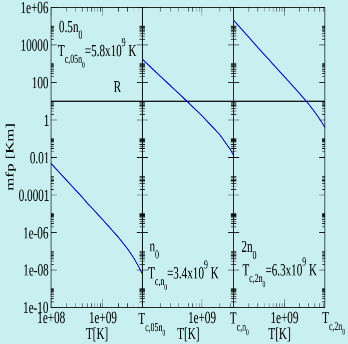
- In normal matter ($T_{cp} = 0$) κ_n dominates over $\kappa_{e\mu}$ at $T \leq 2 \times 10^9$ K.
- $T < T_{cp}$ proton superconductivity sets in $\Rightarrow$ $\kappa_{e\mu}$ starts to grow up much quicker than κ_n ($\kappa_{e\mu} \propto \Delta \propto T_{cp}$) and becomes comparable to or larger than κ_n.
- For a stronger superconductivity with $T_{cp} \gg 9 \times 10^9 K$ $\Rightarrow$ $\kappa_{e\mu}$ dominates over κ_n at any T.
- $10^{25} \lesssim \kappa_{ph} \lesssim 10^{32}$ erg cm^{-1} s^{-1} K^{-1} from 0.5 n_0 to 2 n_0 $\Rightarrow$ Thermal conductivity in the neutron star core is dominated by phonon-phonon collisions.

P. S. Shternin et. al Phys. Rev. D 75, 103004 (2007)

Thermal conductivity mean free path of the phonons

Thermal conductivity mean free path of the phonons $\Rightarrow l = \frac{\kappa}{\frac{1}{3} c_v c_s}$

heat capacity for phonons $\Rightarrow c_v = \frac{2\pi^2}{15 c_s^3} T^3$



- κ_{phn} is temperature independent, $c_v \propto T^3$. Temperature dependence of mfp $\Rightarrow l \propto 1/T^3$.
- Superfluid phonon mfp stays below the radius of the star
  - $n = 0.5 n_0$
  - $n = n_0, \ T \geq 6 \times 10^8 K$
  - $n = 2n_0, \ T \geq 3 \times 10^9 K$

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Summary

- The cooling of a neutron star depends on the rate of neutrino emission and the thermal conductivity.
- Thermal conductivity $\Rightarrow$ solving the Boltzmann equation.
- κ vanishes $\Rightarrow$ linear phonon dispersion law. We calculate first correction in dispersion relation which depends on the gap of neutron matter.
- Phonon dispersion law curves downward beyond linear order $\Rightarrow$ collisional processes of $1 \rightarrow 2$ kinematically forbidden.
- $\kappa \Rightarrow$ phonon scattering rates $\Rightarrow$ effective field theory techniques in terms of the APR EoS of the system.
- $\kappa \propto 1/\Delta^6$ the factor of proportionality depends on the density and EoS of the superfluid.
- Thermal conductivity in the neutron star core is dominated by phonon-phonon collisions when phonons are in a pure hydrodynamical regime.

THANK YOU

continued ...

- Phonon velocity $\Rightarrow$

$$v_{\mathrm{ph}} = \sqrt{\frac{\partial P}{\partial \tilde{\rho}}} \equiv c_s$$

  - $\tilde{\rho} \Rightarrow$ mass density, related to the particle density $(\rho) \Rightarrow \tilde{\rho} = m\rho$.

- Three phonon self-coupling constants $\Rightarrow$

$$g = \frac{1}{6\sqrt{m\rho}\ c_s}\left(1 - 2\frac{\rho}{c_s}\frac{\partial c_s}{\partial \rho}\right) \ , \qquad \eta_g = \frac{c_s}{6\sqrt{m\rho}\ g}$$

- Four phonon coupling constants

$$\lambda = \frac{1}{24\ m\rho\ c_s^2}\left(1 - 8\frac{\rho}{c_s}\frac{\partial c_s}{\partial \rho} + 10\frac{\rho^2}{c_s^2}\left(\frac{\partial c_s}{\partial \rho}\right)^2 - 2\frac{\rho^2}{c_s}\frac{\partial^2 c_s}{\partial \rho^2}\right) \ ,$$

$$\eta_{\lambda_2} = \frac{c_s^2}{8\ m\rho\ \lambda} \ , \qquad \eta_{\lambda,1} = 2\frac{\eta_{\lambda,2}}{\eta_g}$$

EoS for superfluid neutron star matter

- Speed of sound at $T = 0$ and the different phonon selfcouplings $\Rightarrow$ the EoS for neutron matter in neutron stars.
- A common benchmark for nucleonic EoS is APR98
  Akmal, Pandharipande and Ravenhall Phys. Rev. C 58, 1804 (1998)
- Later parametrized $\Rightarrow$ H. Heiselberg, M. Hjorth-Jensen, Phys. Rep. 328, 237-327 (2000)

$$
\begin{aligned}
E/A &= \mathcal{E}_0 u \frac{u - 2 - \delta}{1 + \delta u} + S_0 u^{\gamma} (1 - 2x_p)^2 \\
u &= \rho/\rho_0 \qquad \mathcal{E}_0 = 15.8 \text{ MeV} \\
x_p &= \rho_p/\rho_0 \qquad S_0 = 32 \text{ MeV} \\
\delta &= 0.2 \qquad\quad \gamma = 0.6 \\
\rho_0 &= 0.16 \text{ fm}^{-3}
\end{aligned}
$$

- For stable matter made up of neutrons, protons and electrons c_s at $T = 0$ is

$$
c_s(\rho, x_p) \approx \sqrt{\frac{1}{m} \frac{\partial P_N(\rho, x_p)}{\partial \rho_n}}
$$

Gap parameter

- ${}^1S_0$ and averaged ${}^3P_2$ neutron gaps
- Energy gap (Fermi surface) by the phenomenological formula

$$\Delta(k_F) = \Delta_0 \frac{(k_F - k_1)^2}{(k_F - k_1)^2 + k_2} \frac{(k_F - k_3)^2}{(k_F - k_3)^2 + k_4}$$

$\underline{{}^1S_0(A) + {}^3P_2(i),\ {}^1S_0(c) + {}^3P_2(k)}$

model	Δ_0 (Mev)	k_1 (fm^{-1})	k_2 (fm^{-1})	k_3 (fm^{-1})	k_4 (fm^{-1})
A	9.3	0.02	0.6	1.55	0.32
c	22	0.3	0.09	1.05	4
i	10.2	1.09	3	3.45	2.5
k	0.425	1.1	0.5	2.7	0.5

Table : N. Andersson *et. al*, Nucl. Phys. A763, 212-229 (2005).

Model A is for the bare interaction and is relevant in a pure neutron (proton) medium

c is for the ${}^1S_0$ neutron pairing, i, k are for the ${}^3P_2$ neutron channel

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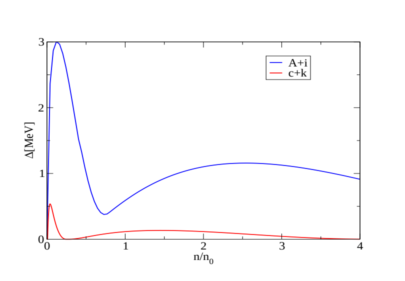

$^1S_0(A)+^3P_2(i)$, $^1S_0(c)+^3P_2(k)$

- $^1S_0(A) \Rightarrow$ maximum gap of about 3 MeV at $p_F \approx 0.85\mathrm{fm}^{-1}$
- $^3P_2(i)$ neutron angular averaged $\Rightarrow$ maximum value for the gap of approximately 1 MeV.
- $^1S_0(c) \Rightarrow$ corrections to the bare nucleon-nucleon potential.
- $^3P_2(k)$ parametrization assuming weak neutron superfluidity in the core with maximum value for the gap of the order of 0.1 MeV.

Validity of result

- Close to T_c higher order corrections in the energy and momentum expansion should be taken into account in both the phonon dispersion law and self-interactions.
- Density of superfluid phonons becomes very dilute at very low $T \Rightarrow$ difficult to maintain a hydrodynamical description of their behavior.
- Phonons would behave in the low-T regime ballistically.
- Thermal conductivity due to phonons would be then dominated by the collisions of the phonons with the boundary $\Rightarrow \kappa = \frac{1}{3} c_v c_s R$

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