QCD Thermodynamics up to three loop at finite T and μ

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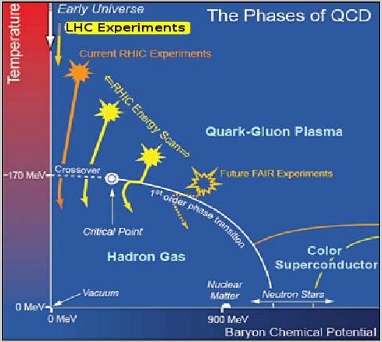
QCD Phase Diagram & EoS

- At High temperature and/or density Quarks and Gluons become deconfined and produce QGP.

- In ongoing RHIC experiments and and also future FAIR experiments the chemical potential of deconfined Nuclear matter is finite.

- Determination of EoS of hot and dense Nuclear matter is essential to QGP phenomenology.



Thermodynamics using Lattice QCD

- The currently most reliable method for determining the equation of state at finite temperature is lattice QCD.
- Due to the sign problem, lattice QCD can not compute EoS at finite baryon chemical potential straightforwardly.
- It can compute thermodynamic functions at small chemical potential by making a Taylor expansion of the partition function around $\mu = 0$ and extrapolating the result as

$$P(T,\mu) = P(T,\mu=0) + \frac{\mu^2}{2}\,\frac{\partial^2 P}{\partial \mu^2}\bigg|_{\mu=0} + \frac{\mu^4}{4!}\,\frac{\partial^4 P}{\partial \mu^4}\bigg|_{\mu=0} + \cdots$$

- The extrapolations can only be trusted at small chemical potential, it would be nice to have an alternative framework for calculating QCD thermodynamical quantities at finte T and μ.

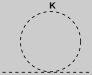
Thermodynamics using perturbation theory

- At sufficiently high temperature, the value of the strong coupling constant is small $\Rightarrow$ It works well at high T.
- Unfortunately, it turns out that a strict expansion in the coupling constant does not converges at the temperature those are relevant for heavy-ion collision experiments.
- The source of the poor convergence comes from contributions from soft momenta, $p \sim gT$.
- One needs a way of reorganizing the perturbative series which treats the soft sector more carefully.

Hard Thermal Loop perturbation theory

- Hard Thermal Loop (HTL) perturbation theory is a gauge invariant reorganization of usual perturbation at finite temperature and finite chemical potential and higher order diagrams contribute to lower order.
- In HTL approximation we define *Two Scales of Momentum*
  1. Hard momentum: $p_0, p \sim T$.
  2. Soft momentum: $p_0, p \sim gT$.
- In HTL approximation we are interested in high temperature limits, so one can take Loop Momentum $\gg$ External Momentum

Scalar case:

$\Pi_1 =$ 

$= g^2T^2.$

Effective propagator $D^\star = \frac{1}{P^2 - \Pi_1}$

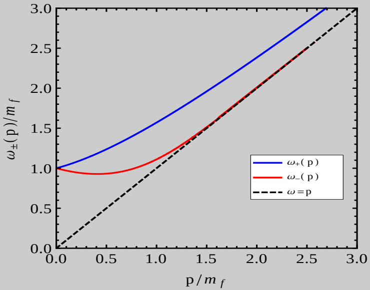
HTL in gauge theory: Quark Propagator

Quark propagator:

$$iS^{\star}(P) = \frac{1}{2}\left[\frac{\gamma^0 - \vec{\gamma}\cdot\hat{p}}{D_+(P)} + \frac{\gamma^0 + \vec{\gamma}\cdot\hat{p}}{D_-(P)}\right].$$

$$D_{\pm}(p_0, p) = -p_0 \pm p + \frac{m_f^2}{p}\left[\pm 1 + \frac{1}{2}\left(1 \mp \frac{p_0}{p}\right)\ln\frac{p_0+p}{p_0-p}\right]$$

Dispersion relation: $D_{\pm}(p_0, p) = 0$



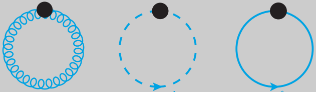
- Total Lagrangian density:

$$
\begin{aligned}
\mathcal{L} &= \left.\left(\mathcal{L}_{\rm QCD} + \mathcal{L}_{\rm HTL}\right)\right|_{g\to\sqrt{\delta}g} + \Delta\mathcal{L}_{\rm HTL}, \\
\mathcal{L}_{\rm HTL} &= (1-\delta) i m_q^2 \bar{\psi}\gamma^\mu \left\langle \frac{Y_\mu}{Y\cdot D} \right\rangle_{\hat{\mathbf{y}}} \psi \\
&\quad -\frac{1}{2}(1-\delta) m_D^2 \,\mathrm{Tr}\left(F_{\mu\alpha} \left\langle \frac{Y^\alpha Y_\beta}{(Y\cdot D)^2} \right\rangle_{\hat{\mathbf{y}}} F^{\mu\beta} \right),
\end{aligned}
$$

- The HTLpt Lagrangian reduces to the QCD Lagrangian if we set $\delta = 1$.
- Physical observables are calculated in HTLpt by expanding in powers of δ, truncating at some specified order, and then setting $\delta = 1$.
- m_D and m_q are two parameters will be treated as Debye mass and thermal quark mass respectively.

One loop HTL thermodynamics

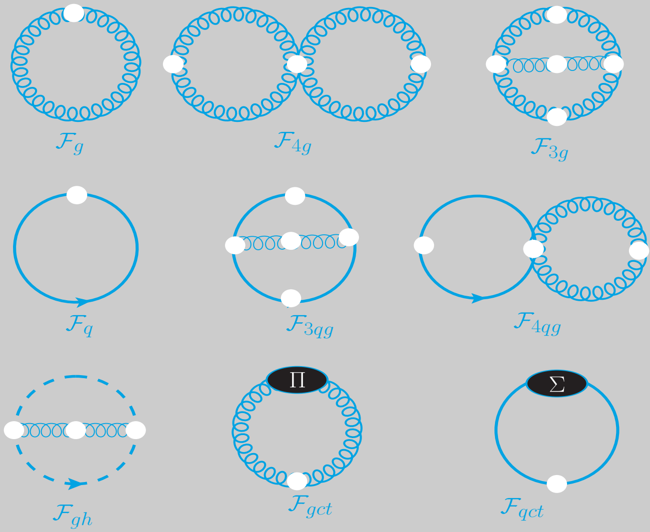
The Feynman diagrams that will contribute to the thermodynamic potential in one loop:



$$
\begin{aligned}
\mathcal{P}(T,\mu) \quad = \quad & 2N_f N_c T \int \frac{d^3k}{(2\pi)^3} \Bigg[\ln\left(1+e^{-\beta(\omega_+-\mu)}\right) + \ln\left(\frac{1+e^{-\beta(\omega_--\mu)}}{1+e^{-\beta(k-\mu)}}\right) \\
& + \ln\left(1+e^{-\beta(\omega_++\mu)}\right) + \ln\left(\frac{1+e^{-\beta(\omega_-+\mu)}}{1+e^{-\beta(k+\mu)}}\right) + \beta\omega_+ + \beta(\omega_- - k) \\
& + \int_{-k}^{k} d\omega \left(\frac{2m_q^2}{\omega^2-k^2}\right) \beta_+(\omega,k) \left[\ln\left(1+e^{-\beta(\omega-\mu)}\right) + \ln\left(1+e^{-\beta(\omega+\mu)}\right) + \beta\omega\right] \Bigg] \\
& + \quad \text{Gluonic contibution}
\end{aligned}
$$

$$
\rho = \frac{\partial \mathcal{P}}{\partial \mu}; \qquad S = \frac{\partial \mathcal{P}}{\partial T}; \qquad \chi = \frac{\partial^2 \mathcal{P}}{\partial \mu^2}
$$

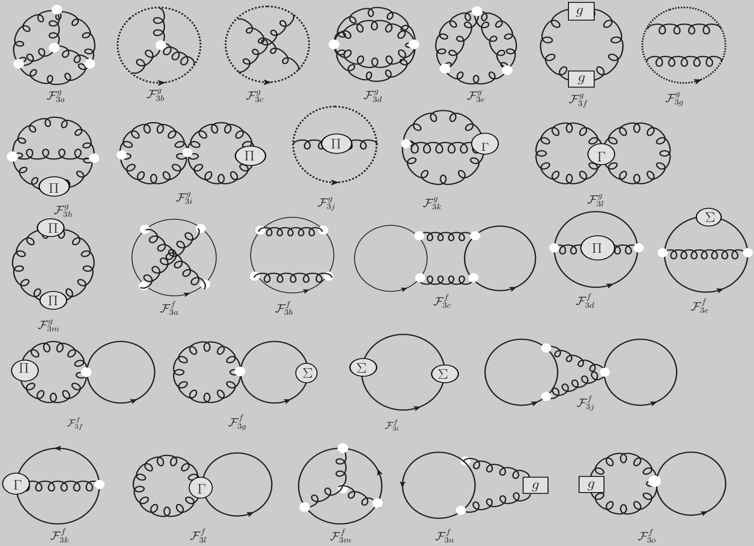
Two loop HTL thermodynamics



NH, Mustafa, Strickland, 1212.1797 & 1302.3228

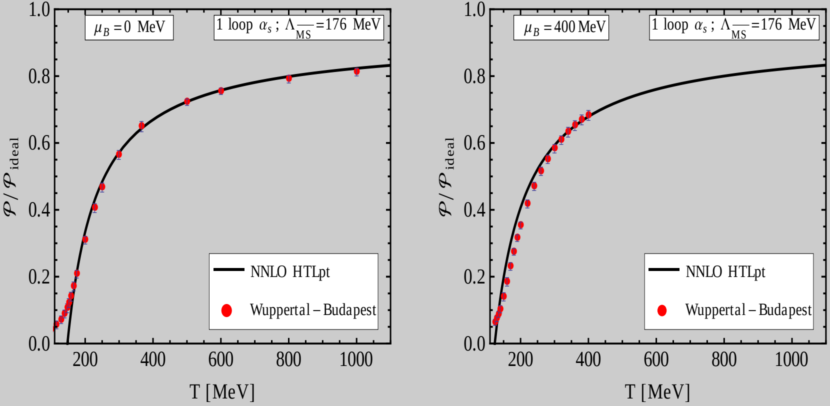
$$
\begin{aligned}
&\mathcal{P}_{\rm NLO}(T,\mu) = \\
&d_A \frac{\pi^2 T^4}{45}\Bigg\{1+\frac{7}{4}\frac{d_F}{d_A}\left(1+\frac{120}{7}\hat{\mu}^2+\frac{240}{7}\hat{\mu}^4\right)-15\hat{m}_D^3-\frac{45}{4}\left(\log\frac{\hat{\Lambda}}{2}-\frac{7}{2}+\gamma+\frac{\pi^2}{3}\right)\hat{m}_D^4 \\
&+60\frac{d_F}{d_A}\left(\pi^2-6\right)\hat{m}_q^4+\frac{\alpha_s}{\pi}\Bigg[15\left(c_A+s_F(1+12\hat{\mu}^2)\right)\hat{m}_D-\frac{5}{4}\left(c_A+\frac{5}{2}s_F\left(1+\frac{72}{5}\hat{\mu}^2+\frac{144}{5}\hat{\mu}^4\right)\right) \\
&-\frac{55}{4}\Bigg\{c_A\left(\log\frac{\hat{\Lambda}}{2}-\frac{36}{11}\log\hat{m}_D-2.001\right)-\frac{4}{11}s_F\Bigg[\left(\log\frac{\hat{\Lambda}}{2}-2.337\right) \\
&+\ (24-18\zeta(3))\left(\log\frac{\hat{\Lambda}}{2}-15.662\right)\hat{\mu}^2+120\left(\zeta(5)-\zeta(3)\right)\left(\log\frac{\hat{\Lambda}}{2}-1.5264\right)\hat{\mu}^4\Bigg]\Bigg\}\hat{m}_D^2 \\
&-45\,s_F\left\{\log\frac{\hat{\Lambda}}{2}+2.198-44.953\hat{\mu}^2-\left(288\ln\frac{\hat{\Lambda}}{2}+19.836\right)\hat{\mu}^4\right\}\hat{m}_q^2 \\
&+\frac{165}{2}\left\{c_A\left(\log\frac{\hat{\Lambda}}{2}+\frac{5}{22}+\gamma\right)-\frac{4}{11}s_F\left(\log\frac{\hat{\Lambda}}{2}-\frac{1}{2}+\gamma+2\ln 2-7\zeta(3)\hat{\mu}^2+31\zeta(5)\hat{\mu}^4\right)\right. \\
&+15s_F\left(2\frac{\zeta'(-1)}{\zeta(-1)}+2\ln\hat{m}_D\right)\left[(24-18\zeta(3))\hat{\mu}^2+120(\zeta(5)-\zeta(3))\hat{\mu}^4\right]\hat{m}_D^3+180\,s_F\hat{m}_D\hat{m}_q^2\Bigg]
\end{aligned}
$$

Three loop HTL thermodynamics



$$
\begin{aligned}
\mathcal{P}_{\rm NNLO} = & \frac{d_A\pi^2T^4}{45}\Bigg[\!\!\Bigg[1+\frac{7}{4}\frac{d_F}{d_A}\left(1+\frac{120}{7}\hat\mu^2+\frac{240}{7}\hat\mu^4\right)-\frac{15}{4}\hat m_D^3-\frac{s_F\alpha_s}{\pi}\left[\frac{5}{8}\left(5+72\hat\mu^2+144\hat\mu^4\right)\right.\\
&+90\hat m_q^2\hat m_D-\frac{15}{2}\left(1+12\hat\mu^2\right)\hat m_D-\frac{15}{2}\left(2\ln\frac{\hat\Lambda}{2}-1-\aleph(z)\right)\hat m_D^3\Big]+s_{2F}\left(\frac{\alpha_s}{\pi}\right)^2\Big[-\frac{45}{2}\hat m_D\left(1+12\hat\mu^2\right)\\
&+\frac{15}{64}\Big\{35-32\left(1-12\hat\mu^2\right)\frac{\zeta'(-1)}{\zeta(-1)}+472\hat\mu^2+1328\hat\mu^4+64\Big(6(1+8\hat\mu^2)\aleph(1,z)+3i\hat\mu(1+4\hat\mu^2)\aleph(0,z)\\
&-36i\hat\mu\aleph(2,z)\Big)\Big\}\Big]+\left(\frac{s_F\alpha_s}{\pi}\right)^2\Big[\frac{5}{4\hat m_D}\left(1+12\hat\mu^2\right)^2+30\left(1+12\hat\mu^2\right)\frac{\hat m_q^2}{\hat m_D}+\frac{25}{12}\Big\{\frac{1}{20}\left(1+168\hat\mu^2+2064\hat\mu^4\right)\\
&+\left(1+\frac{72}{5}\hat\mu^2+\frac{144}{5}\hat\mu^4\right)\ln\frac{\hat\Lambda}{2}+\frac{3\gamma_E}{5}\left(1+12\hat\mu^2\right)^2-\frac{8}{5}(1+12\hat\mu^2)\frac{\zeta'(-1)}{\zeta(-1)}-\frac{34}{25}\frac{\zeta'(-3)}{\zeta(-3)}-\frac{72}{5}\Big[3\aleph(3,2z)\\
&+8\aleph(3,z)-12\hat\mu^2\aleph(1,2z)-2(1+8\hat\mu^2)\aleph(1,z)+12i\hat\mu\left(\aleph(2,z)+\aleph(2,2z)\right)-i\hat\mu(1+12\hat\mu^2)\aleph(0,z)\Big]\Big\}\\
&-\frac{15}{2}\left(1+12\hat\mu^2\right)\left(2\ln\frac{\hat\Lambda}{2}-1-\aleph(z)\right)\hat m_D\Big]+\frac{c_A\alpha_s}{3\pi}\frac{s_F\alpha_s}{\pi}\Big[\frac{15}{2\hat m_D}\left(1+12\hat\mu^2\right)+90\frac{\hat m_q^2}{\hat m_D}\\
&-\frac{235}{16}\Big\{\left(1+\frac{792}{47}\hat\mu^2+\frac{1584}{47}\hat\mu^4\right)\ln\frac{\hat\Lambda}{2}-\frac{24\gamma_E}{47}\left(1+12\mu^2\right)+\frac{319}{940}\left(1+\frac{2040}{319}\hat\mu^2+\frac{38640}{319}\hat\mu^4\right)-\frac{268}{235}\frac{\zeta'(-3)}{\zeta(-3)}\\
&-\frac{144}{47}\left(1+12\hat\mu^2\right)\ln\hat m_D-\frac{44}{47}\left(1+\frac{156}{11}\hat\mu^2\right)\frac{\zeta'(-1)}{\zeta(-1)}-\frac{72}{47}\Big[4i\hat\mu\aleph(0,z)+\left(5-92\hat\mu^2\right)\aleph(1,z)+144i\hat\mu\aleph(2,z)\\
&+52\aleph(3,z)\Big]\Big\}+\frac{315}{4}\Big\{\left(1+\frac{132}{7}\hat\mu^2\right)\ln\frac{\hat\Lambda}{2}+\frac{11}{7}\left(1+12\hat\mu^2\right)\gamma_E+\frac{9}{14}\left(1+\frac{132}{9}\hat\mu^2\right)+\frac{2}{7}\aleph(z)\Big\}\hat m_D\Big]\\
&+\frac{c_A\alpha_s}{3\pi}\Big[-\frac{15}{4}+\frac{45}{2}\hat m_D-\frac{135}{2}\hat m_D^2-\frac{495}{4}\left(\ln\frac{\hat\Lambda_g}{2}+\frac{5}{22}+\gamma_E\right)\Big]+\left(\frac{c_A\alpha_s}{3\pi}\right)^2\Big[\frac{45}{4\hat m_D}-\frac{165}{8}\Big(\ln\frac{\hat\Lambda_g}{2}\\
&-\frac{72}{11}\ln\hat m_D-\frac{84}{55}-\frac{6}{11}\gamma_E-\frac{74}{11}\frac{\zeta'(-1)}{\zeta(-1)}+\frac{19}{11}\frac{\zeta'(-3)}{\zeta(-3)}\Big)+\frac{1485}{4}\left(\ln\frac{\hat\Lambda_g}{2}-\frac{79}{44}+\gamma_E-\ln 2-\frac{\pi^2}{11}\right)\hat m_D\Big]\Bigg]\!\!\Bigg]
\end{aligned}
$$

NNLO pressure for QCD HTL perturbation theory



- Thick Black Line: Renormalization Scale $\Lambda = 2\pi\sqrt{T^2 + \mu^2/\pi^2}$
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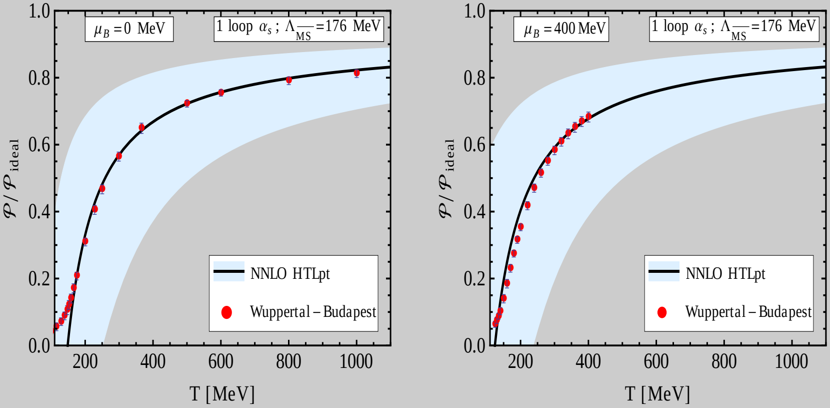
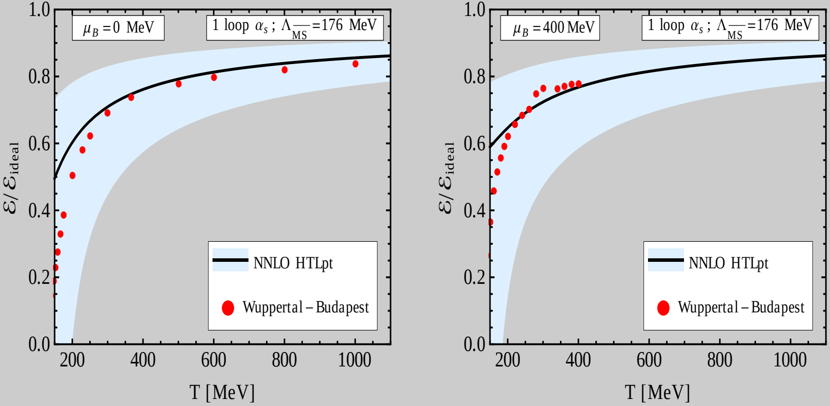
NNLO pressure for QCD HTL perturbation theory



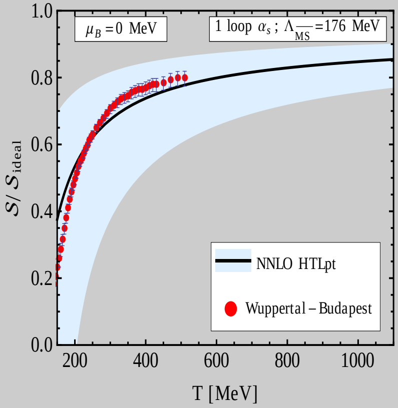
- Thick Black Line: Renormalization Scale $\Lambda = 2\pi\sqrt{T^2 + \mu^2/\pi^2}$
- Band : Varying center value by factor of 2.

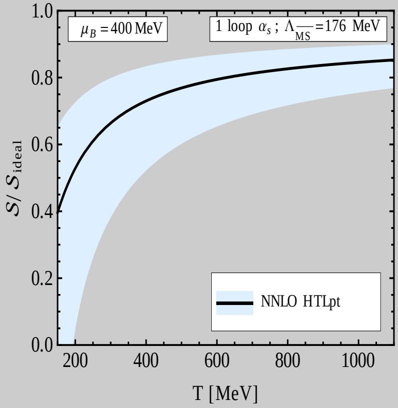
Energy Density



Lattice data have been extracted using: $\mathcal{E} = 3\mathcal{P} + I$

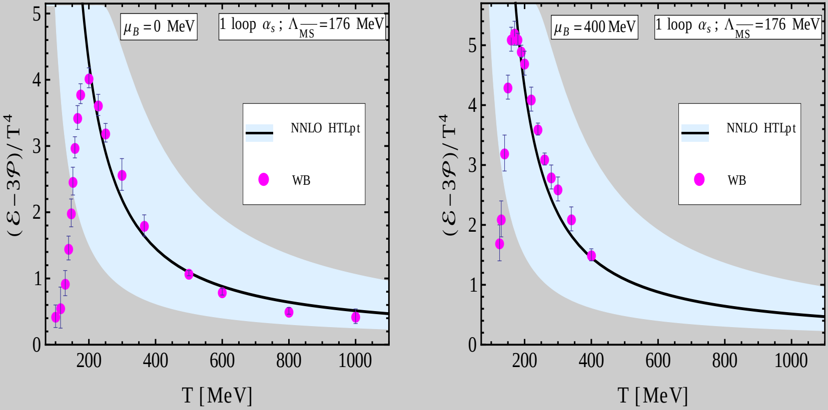
Entropy Density





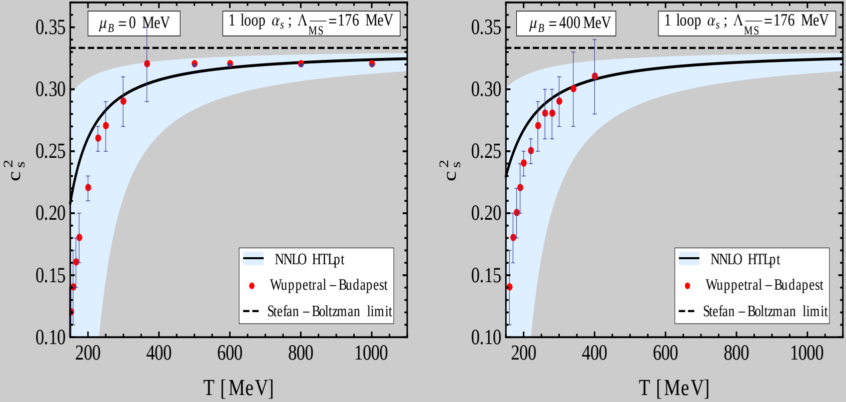
Trace Anomaly

Trace Anomaly$=\mathcal{E}-3\mathcal{P}$



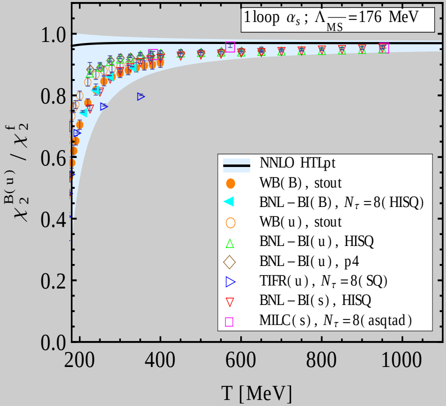
Speed of Sound

$$c_s^2 = \frac{\partial \mathcal{P}}{\partial \mathcal{E}}$$



Second order Quark Number Susceptibility

$$\chi_2^u = \left.\frac{\partial^2 \mathcal{P}}{\partial \mu^2}\right|_{\mu \to 0}$$

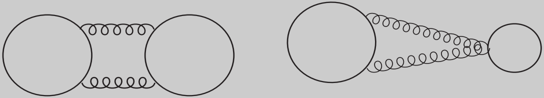


Fourth order Quark Number Susceptibility

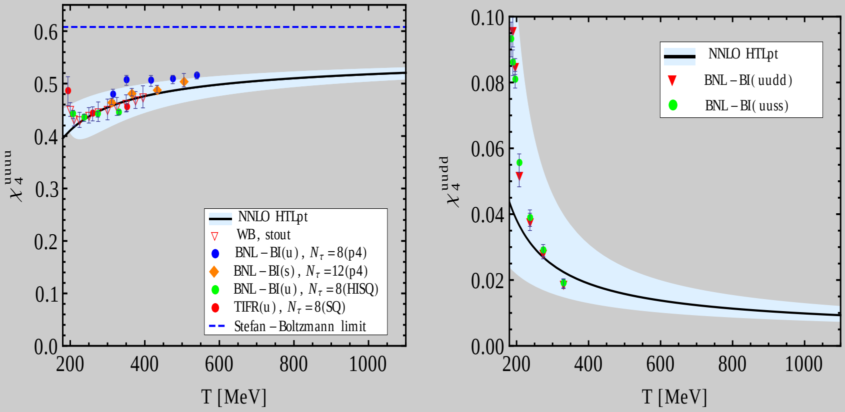
Diagonal Susceptibility $\chi_4^u = \frac{\partial^4 \mathcal{P}}{\partial \mu^4}$

Off-diagonal Susceptibility $\chi_4^{uudd} = \frac{\partial^4 \mathcal{P}}{\partial \mu_u^2 \partial \mu_d^2}$.

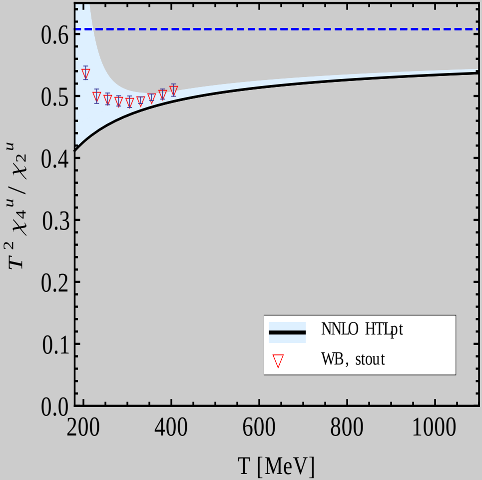
The following two diagrams will contribute to only off-diagonal Susceptibility χ_4^{uudd}.



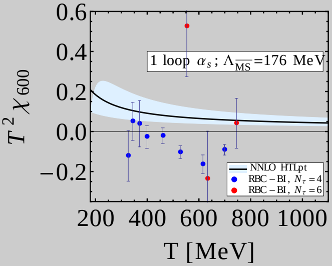
Fourth order Quark Number Susceptibility

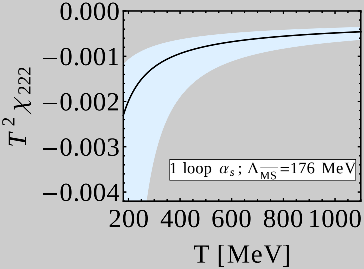


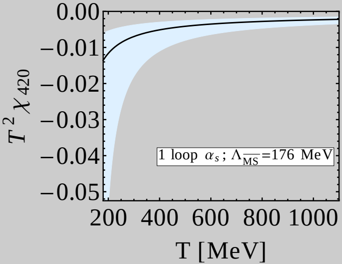
Fourth/second order Quark Number Susceptibility



Sixth order Quark Number Susceptibility







Baryon number susceptibilities

$$\chi_B^n(T) \equiv \left.\frac{\partial^n \mathcal{P}}{\partial \mu_B^n}\right|_{\mu_B=0} .$$

$$\chi_2^B = \frac{1}{9}\left[\chi_2^{uu} + \chi_2^{dd} + \chi_2^{ss} + 2\chi_2^{ud} + 2\chi_2^{ds} + 2\chi_2^{us}\right] ,$$

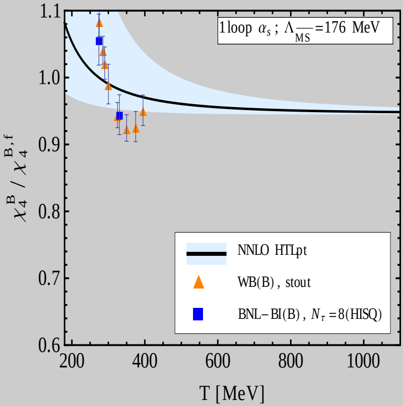
and

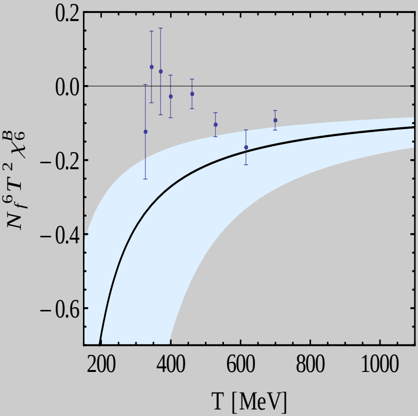
$$\begin{aligned}\chi_4^B \;=\; & \frac{1}{81}\Big[\chi_4^{uuuu} + \chi_4^{dddd} + \chi_4^{ssss} + 4\chi_4^{uuud} + 4\chi_4^{uuus} \\ & + 4\chi_4^{dddu} + 4\chi_4^{ddds} + 4\chi_4^{sssu} + 4\chi_4^{sssd} + 6\chi_4^{uudd} \\ & + 6\chi_4^{ddss} + 6\chi_4^{uuss} + 12\chi_4^{uuds} + 12\chi_4^{ddus} + 12\chi_4^{ssud}\Big] .\end{aligned}$$

For $\mu_u = \mu_d = \mu_s = \mu_B/3$,

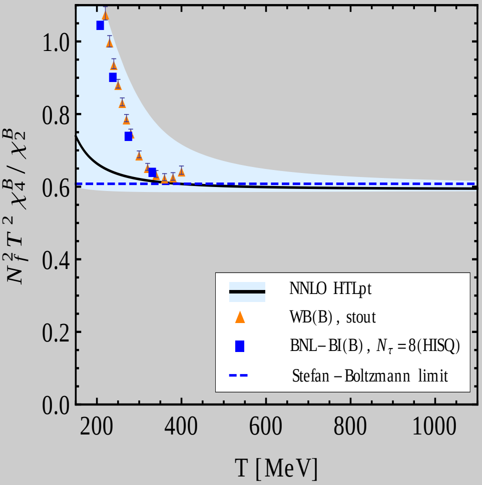
$$\chi_2^B = \frac{1}{3}\chi_2^{uu} \qquad \chi_4^B = \frac{1}{27}\left[\chi_4^{uuuu} + 6\chi_4^{uudd}\right]$$

Baryon number susceptibilities





Fourth/second order Baryon Number Susceptibility

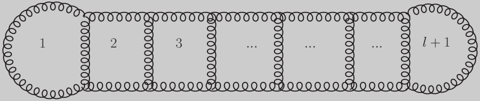


Conclusions

- I have discussed about thermodynamic quantities in leading as well as beyond leading order using HTLpt.
- Thermodynamical potential produce correct perturbative order upto g, g^3 and g^5 if one expands for small g in case of one loop, two loop and three loop respectively.
- NNLO pressure is completely analytic and does not depend on any free parameter except the choice of the renormalization scale.
- For three loop case, we found good agreement between our results and LQCD results down to temperature ~ 250 MeV.

Back up Slides

Linde's Problem



Gluon propagator$=\sum_n \frac{1}{\omega_n^2+k^2+m^2}$ $\qquad\omega_n = 2\pi n T$

$m =$ some screening mass.

The leading infra-red can be estimated by power counting in partition function as

$$Z_l \sim g^{2l}\left(T\int d^3k\right)^{l+1} k^{2l}(k^2+m^2)^{-3l}$$

- $l < 3$: Z_l is IR regular.
- $l = 3$: $Z_l \sim g^6 T^4 \log\left(\frac{T}{m}\right)$
- $l > 3$: $Z_l \sim g^6 T^4 \left(\frac{g^2 T}{m}\right)^{l-3}$

Linde's Problem

- For longitudinal gluons, the screening mass $m_{el} \sim gT$. So for $l > 3$, $Z_l \sim g^{l+3}T^4$.
- For transverse gluons, the screening mass $m_{mag} \sim g^2T$, So for $l > 3$, $Z_l \sim g^6T^4$.

Which is a complete failure of Perturbation theory.

Running coupling

QCD running coupling:

$$\alpha_s(\Lambda) \;=\; \frac{g(\Lambda)^2}{4\pi} \;=\; \frac{12\pi}{(11N_c - 2N_f)\log\left(\Lambda^2/\Lambda^2_{\overline{\rm MS}}\right)}\,, \tag{2}$$

- $\Lambda \rightarrow$ Renormalization scale.
- The middle line corresponds to $\Lambda = 2\pi\sqrt{T^2 + \mu^2/\pi^2}$.
- We fix the QCD scale $\Lambda_{\overline{\rm MS}}$ by requiring that $\alpha_s(1.5\text{GeV}) = 0.326$ which is obtained from lattice measurements *A. Bazavov et al., Phys. Rev. D 86 (2012) 114031.*
- $\alpha_s(1.5\text{GeV}) = 0.326 \Longrightarrow \Lambda_{\overline{\rm MS}} = 176$ MeV for one loop α_s.
  $\Lambda_{\overline{\rm MS}} = 316$ MeV for three loop α_s.

The value of $\Lambda_{\rm middle}$

Matsubara frequency: $\omega_n^b = 2n\pi T$ for boson,

$$\omega_n^f = (2n+1)\pi T + i\mu \quad \text{for fermion.}$$

At $\mu = 0$, $\Lambda_{\rm middle} = \omega_1^b = 2\omega_0^f = 2\pi T$.

At $\mu \neq 0$, $\Lambda_{\rm middle} = 2|\omega_0^f| = 2\pi\sqrt{T^2 + \mu^2/\pi^2}$.

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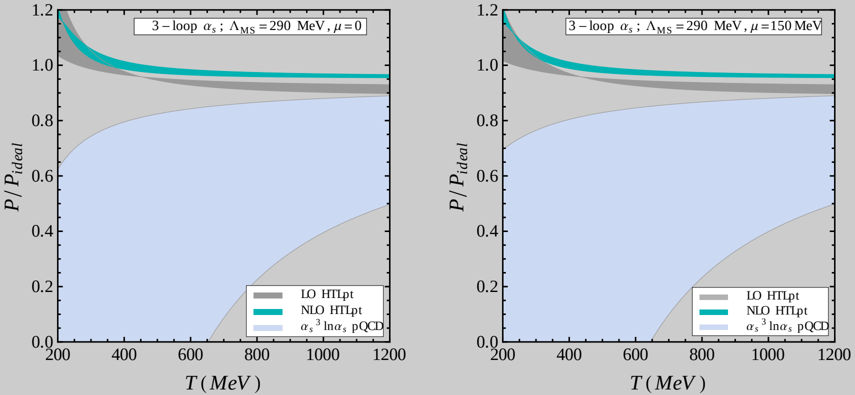

Quark Number Susceptibility

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