Bulk viscosity of Hadronic Resonance Gas

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Outline

- **Transport co-efficients?**
- **How to estimate?**
- **Bulk viscosity from thermodynamics: Calculations**
- **Results**

Transport Coefficients: in general sense

Effect = −Response of system ⊗ Cause

For response originating from systems tendency to maintain equilibrium

$$\text{Dissipative current} = - \text{System's response to maintain equilibrium}$$
$$\otimes \quad \text{Measure of devation from eqlibrium}$$

$$\text{Dissipative current} = - \sum_{\text{sources}} \text{Green's function} \otimes \text{Source}$$

Transport co-efficients for Hydrodynamic system

- **Hydrodynamic system defined with two thermodynamic variable + flow velocity vector**
- **Deviation from equilibrium: e.g. $\longrightarrow$ space gradient of these quantities that is not allowed by equilibrium condition**
- **Dissipative current: proportional to system's response and strength of gradient.**
- **Response of the system under such space gradients $\longrightarrow$ transport co-efficients**
- **Response depends on interaction strength & ' nature of phase space distribution'**

- **For gradient of fluid velocity which is not allowed by equilibrium situation, there will be system's response to bring it back to equilibrium.**
- **Velocity is a vector, it has three component, each component can have gradient in all directions of space. But we can categories these gradients into two classes.**
  - **gradients along the direction which is same as the corresponding component:** $\frac{\partial U_i}{\partial x_i}$; **can be written as** $\delta_{ij} \nabla \cdot \vec{U}$
  - **gradient in a direction which is perpendicular to the direction of the component:** $\frac{\partial U_i}{\partial x_j}$; $i \neq j$; **can be written as** $\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} - \frac{2}{3}\delta^i_j \frac{\partial U_l}{\partial x_l}\right)$
- **Accordingly system has two separate types of response to these two type of gradients. One is shear viscosity, second is bulk viscosity; first one for gradient along perpendicular direction, second one for gradient along same direction, which is measure of compressibility of fluid, incompressible fluid does not allow creation of such divergence.**

Shear and Bulk Viscosity

Energy-momentum tensor is the quantity which characterizes the system

- **Ideal:**

$$T^{\mu}_{\nu} = (\rho + P)U^{\mu}U_{\nu} + Pg^{\mu}_{\nu},$$

$$T^{\mu\nu} = \int d^3p \, \frac{p^{\mu}p^{\nu}}{p^0} f_{eq}(\vec{x}, \vec{p}, t)$$

- **With dissipation**

$$T^{\mu}_{\nu} = (\rho + P)U^{\mu}U_{\nu} + Pg^{\mu}_{\nu} + \text{term for bulk} + \text{term for shear} + T.C.$$

:

$$T^{\mu\nu} = \int d^3p \, \frac{p^{\mu}p^{\nu}}{p^0} \{f_{eq}(\vec{x}, \vec{p}, t) + \delta f(\vec{x}, \vec{p}, t)\}$$

- **Term for shear:**

$$\Sigma^{i}_{j}(\vec{x}, t) = -\eta\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} - \frac{2}{3}\delta^{i}_{j}\frac{\partial U_l}{\partial x_l}\right).$$

- **Term for bulk:**

$$-\zeta\frac{1}{3}\delta_{ij}\nabla \cdot \vec{U} \equiv \delta P \delta_{ij}$$

Using Boltzmann Eq.: Steps are following

- **Find δf**

$$\frac{\partial f}{\partial t} + \frac{\vec{p}}{\epsilon}\vec{\nabla} f = -\frac{\delta f}{\tau}$$

$$\delta f = -\tau\{\frac{\partial f}{\partial t} + \frac{\vec{p}}{\epsilon}\vec{\nabla} f\}$$

- **Write the right hand side in terms of $(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} - \frac{2}{3}\delta^i_j\frac{\partial U_l}{\partial x_l})$ and $\delta_{ij}\nabla\cdot\vec{U}$ as basis,**
  - **Use continuity and conservation equations and thermodynamic relations,**
  - **With form of f to that of equilibrium form but involving T,μ,U having time varying space profile with gradients.**
  - **U into the distribution through $\epsilon = U.P$.**
- **Then use this δf to get $T^{\mu\nu}$**
- **In this form of $T^{\mu\nu}$, identifying terms with $(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} - \frac{2}{3}\delta^i_j\frac{\partial U_l}{\partial x_l})$ and $\delta_{ij}\nabla\cdot\vec{U}$, we get the bulk and shear viscosities.**

Other methods are

- finding the limiting values of Green's function that connects dissipative current to thermodynamic forces.
- from thermal average of commutators of dissipative terms in energy momentum tensor i.e, using Kubo relation
- bulk viscosity can be calculated from thermodynamic consideration, from systems tendency to maintain equilibrium under change that causes divergence of fluid velocity

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- **as it is related to the dissipative current which involves change in pressure (a thermodynamic quantity) with a δ_{ij} basis.**
- **if we can connect any change to divergence of fluid velocity, finding the response of the system to such changes we get the bulk viscosity of the system. In other word any thing that causes $\vec{\nabla} \cdot \vec{U}$ type deviation can be used to calculate the systems response to it, i.e, bulk viscosity.**

Bulk viscosity from thermodynamic derivatives

- **It can be calculated from system's effort to maintain equilibrium under change of number density.**
- **Because change in number density can change pressure and hence create $\vec{\nabla} \cdot \vec{U}$ at that point.**
- **If the system is slightly away from its equilibrium N then if τ_R is the relaxation time for chemical equilibrium then the rate at which system will try to get back to its equilibrium value, is**

$$\frac{dN}{dt} = -\frac{1}{\tau_R}(N_{eq} - N)$$

Describes response of the system under this change

- **Now if one try to change the number at a rate more than τ_R^{-1} we will not find the system in equilibrium. So if deviation in number δN in out of equilibrium is**

$$\delta N = -\tau_R \frac{dN_{eq}}{dt}$$

then one can use thermodynamic calculations for such changes.

Bulk viscosity from thermodynamic derivatives: Formulae

For slowly varying equilibrium particle number N_{eq} with relaxation time scale τ_{R}, the change in N_{eq}, δN can be written as

$$\delta N = -\tau_{\mathrm{R}} \frac{\partial N_{eq}}{\partial t}$$

ThisδN is expected togive rise to the following change in P, δP

$$\delta P = \left(\frac{\partial P}{\partial n} \right)_{\epsilon} \frac{\delta N}{\Omega}$$

Assuming adiabatically we have $\frac{dS}{dt} = 0$

$$\Rightarrow \frac{\partial s}{\partial t} = -s \nabla \cdot u$$

Bulk viscosity from thermodynamic derivatives: Formulae

Hence, the rate of change of particle number becomes

$$\frac{\partial N_{eq}}{\partial t} = -\Omega s \frac{\partial n}{\partial s} \nabla \cdot u$$

$$\delta P = \left(\frac{\partial P}{\partial n}\right)_{\epsilon} \frac{\partial n}{\partial s} s \tau_{\mathrm{R}} \nabla \cdot u$$

$$\frac{\zeta}{s \tau_{\mathrm{R}}} = \left(\frac{\partial P}{\partial n}\right)_{\epsilon} \frac{\partial n}{\partial s}$$

The differential of $P\left(T,\mu\right)$ can be written as

$$dP = \left(\frac{\partial P}{\partial T}\right) dT + \left(\frac{\partial P}{\partial \mu}\right) d\mu$$

$$\frac{\partial P}{\partial n} = \left(\frac{\partial P}{\partial T}\right) \frac{\partial T}{\partial n} + \left(\frac{\partial P}{\partial \mu}\right) \frac{\partial \mu}{\partial n}$$

$$\left(\frac{\partial P}{\partial n}\right)_{\epsilon} = \left(\frac{\partial P}{\partial T}\right) \left(\frac{\partial T}{\partial n}\right)_{\epsilon} + \left(\frac{\partial P}{\partial \mu}\right) \left(\frac{\partial \mu}{\partial n}\right)_{\epsilon}$$

Using

$$\left(\frac{\partial n}{\partial T}\right)_{\epsilon} = \left(\frac{\partial n}{\partial T}\right) + \left(\frac{\partial n}{\partial \mu}\right) \left(\frac{\partial \mu}{\partial T}\right)_{\epsilon},$$

$$\left(\frac{\partial n}{\partial \mu}\right)_{\epsilon} = \left(\frac{\partial n}{\partial \mu}\right) + \left(\frac{\partial n}{\partial T}\right) \left(\frac{\partial T}{\partial \mu}\right)_{\epsilon}$$

and the constant ϵ trajectory $\left(\frac{\partial T}{\partial \mu}\right)_{\epsilon} = -\frac{\left(\frac{\partial \epsilon}{\partial \mu}\right)}{\left(\frac{\partial \epsilon}{\partial T}\right)}$.

Finally $\frac{\partial n}{\partial s}$ is evaluated as

$$\left(\frac{\partial n}{\partial s}\right) = \left(\frac{\partial n}{\partial T}\right) \left(\frac{\partial T}{\partial s}\right) + \left(\frac{\partial n}{\partial \mu}\right) \left(\frac{\partial \mu}{\partial s}\right)$$

.

Thus, bulk viscosity to entropy density ratio in units of the relaxation time scale becomes

$$\frac{\zeta}{s\tau_{\text{chem}}} = -\left(\frac{\partial P}{\partial n}\right)_{\epsilon}\left(\frac{\partial n}{\partial s}\right)$$

$$\frac{\zeta}{s\tau_{\text{chem}}} = -\left(\frac{\left(\frac{\partial P}{\partial T}\right)}{\left(\frac{\partial n}{\partial T}\right) - \left(\frac{\partial n}{\partial \mu}\right)\frac{\left(\frac{\partial \epsilon}{\partial T}\right)}{\left(\frac{\partial \epsilon}{\partial \mu}\right)}} + \frac{\left(\frac{\partial P}{\partial \mu}\right)}{\left(\frac{\partial n}{\partial \mu}\right) - \left(\frac{\partial n}{\partial T}\right)\frac{\left(\frac{\partial \epsilon}{\partial \mu}\right)}{\left(\frac{\partial \epsilon}{\partial T}\right)}}\right) \times \left(\left(\frac{\partial n}{\partial T}\right)\left(\frac{\partial T}{\partial s}\right) + \left(\frac{\partial n}{\partial \mu}\right)\left(\frac{\partial \mu}{\partial s}\right)\right)$$

Now all the derivatives required above can be evaluated starting from the expression of $\ln Z$ within the HRG model. To begin with, the partition function $Z\,(T, V, \mu)$ is given by

$$\ln\, Z^{GC}(T, V, \{\mu_i\}) = \sum_i \frac{g_i}{2\pi^2} V T^3 \sum_{n=1}^{\infty} \frac{(\mp 1)^{(n+1)}}{n^4} \left(\frac{n m_i}{T}\right)^2 K_2\left(\frac{n m_i}{T}\right) e^{n\beta\mu_i}$$

The pressure P is obtained by operating $T\frac{\partial}{\partial V}$ on $Z^{GC}(T, V, \{\mu_i\})$

$$P^{GC}(T, V, \{\mu_i\}) = \sum_i \frac{g_i}{2\pi^2} T^4 \sum_{n=1}^{\infty} \frac{(\mp 1)^{(n+1)}}{n^4} \left(\frac{n m_i}{T}\right)^2 K_2\left(\frac{n m_i}{T}\right) e^{n\beta\mu_i}$$

Further the derivatives of P are obtained as

$$\left(\frac{\partial P^{GC}}{\partial T}\right) = \frac{1}{V}\left\{ln\, Z^{GC} + \frac{1}{T}\left(E^{GC} - \sum_i \mu_i N_i^{GC}\right)\right\}$$

$$\left(\frac{\partial P^{GC}}{\partial \mu}\right) = \frac{1}{V}\sum_i B_i N_i^{GC}$$

The particle number and its derivatives are given by

$$
\begin{aligned}
N_i^{GC}(T,V,\mu_i) &= T\frac{\partial \ln Z^{GC}}{\mu_i} \\
N_i^{GC}(T,V,\mu_i) &= \frac{g_i}{2\pi^2}VT^3\sum_{n=1}^{\infty}\frac{(\mp 1)^{(n+1)}}{n^3}\left(\frac{nm_i}{T}\right)^2 K_2\left(\frac{nm_i}{T}\right)e^{n\beta\mu_i} \\
\left(\frac{\partial N_i^{GC}}{\partial T}\right) &= \frac{g_i V m_i^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{(\mp 1)^{n+1}}{n}e^{n\mu_i/T}\{\frac{nm_i}{2T}K_1\left(\frac{nm_i}{T}\right) \\
&\quad + \left(1-\frac{n\mu_i}{T}\right)K_2\left(\frac{nm_i}{T}\right)+\frac{nm_i}{2T}K_3\left(\frac{nm_i}{T}\right)\} \\
\left(\frac{\partial N_i^{GC}}{\partial \mu}\right)_T &= \frac{B_i g_i}{2\pi^2}VT^2\sum_{n=1}^{\infty}\frac{(\mp 1)^{(n+1)}}{n^2}\left(\frac{nm_i}{T}\right)^2 K_2\left(\frac{nm_i}{T}\right)e^{n\beta\mu_i}
\end{aligned}
$$

The energy and its derivatives are given by,

$$\begin{aligned} E^{GC}(T,V,\{\mu_i\}) &= T^2\frac{\partial \ln Z^{GC}}{\partial T} + \sum_i \mu_i N_i^{GC} \\ E^{GC}(T,V,\{\mu_i\}) &= \sum_i \frac{g_i}{2\pi^2} V T^2 m_i^2 \sum_{n=1}^{\infty} \frac{(\mp 1)^{(n+1)}}{n^2} e^{n\beta\mu_i} \{ \left(\frac{nm_i}{2T}\right) K_1\left(\frac{nm_i}{T}\right) \\ &+ \left(1-\frac{n\mu_i}{T}\right) K_2\left(\frac{nm_i}{T}\right) + \left(\frac{nm_i}{2T}\right) K_3\left(\frac{nm_i}{T}\right) \} \\ &+ \sum_i \mu_i N_i^{GC} \end{aligned} \tag{1}$$

From that

$$\left(\frac{\partial E^{GC}}{\partial T}\right)_\mu \text{ and } \left(\frac{\partial E^{GC}}{\partial \mu}\right)_T$$

Finally the entropy and its derivatives are given by

$$\begin{aligned} S^{GC}(T,V,\{\mu_i\}) &= \frac{1}{T}\left\{E^{GC}(T,V,\{\mu_i\}) + P^{GC}(T,V,\{\mu_i\})V - \sum_i \mu_i N_i^{GC}\right\} \\ \left(\frac{\partial S^{GC}}{\partial T}\right) &= -\frac{S}{T} + \frac{1}{T}\left\{\left(\frac{\partial E^{GC}}{\partial T}\right) + V\frac{\partial P}{\partial T} - \sum_i \mu_i \left(\frac{\partial N_i^{GC}}{\partial T}\right)\right\} \\ \left(\frac{\partial S^{GC}}{\partial \mu}\right) &= \frac{1}{T}\left\{\left(\frac{\partial E^{GC}}{\partial \mu}\right) + V\frac{\partial P}{\partial \mu} - \sum_i B_i N_i - \sum_i \mu_i \left(\frac{\partial N_i^{GC}}{\partial \mu}\right)\right\} \end{aligned}$$

where we have used

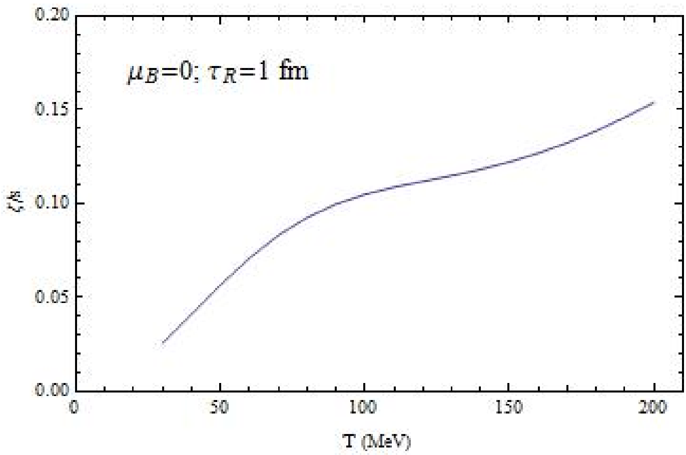
$$\frac{\partial}{\partial \mu} = \sum_i \frac{\partial \mu_i}{\partial \mu}\frac{\partial}{\partial \mu_i}$$

Relevant thermodynamic system: HRG

HRG system Considered:

- **after hadronization**
- **before chemical freeze out**
- **first all possible resonances, from particle data book up to 2.5 GeV are taken into account.**
- **Hagedron density of state included.**

Results and Plots, Comparisons, Explanations



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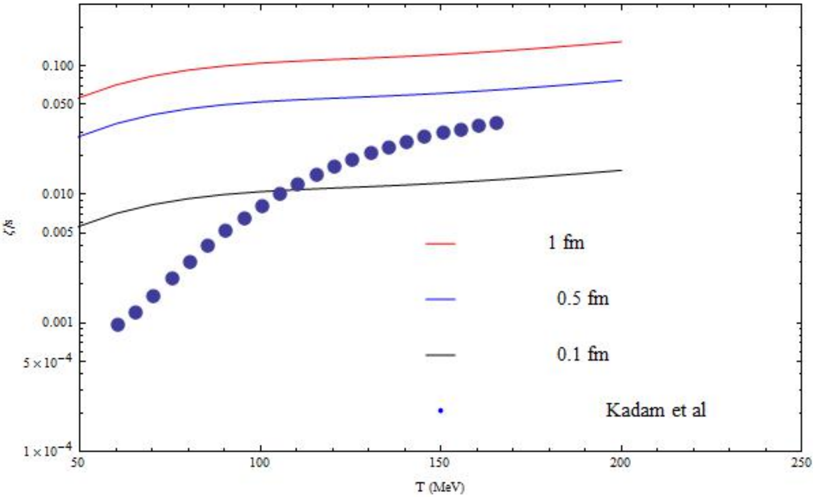

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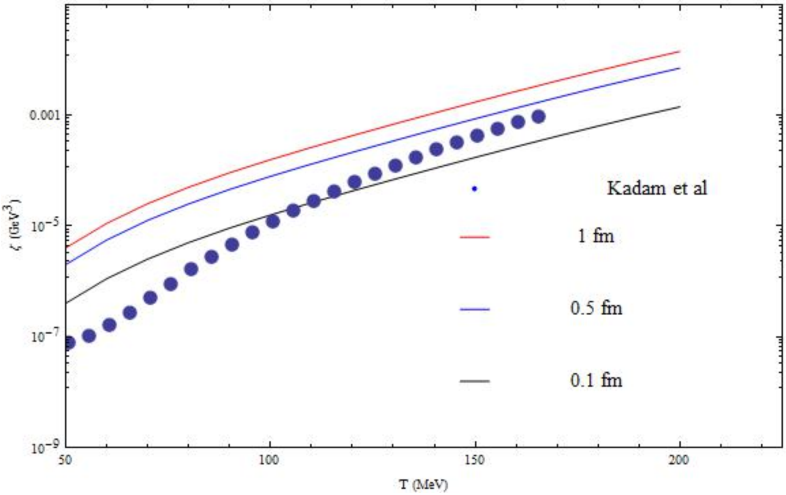


T,μ dependent τ:

- τ_R is estimated from

$$\tau_R = \frac{1}{n\sigma\langle v\rangle} \simeq \frac{\langle\langle E\rangle\rangle}{n\sigma\langle\langle p\rangle\rangle} = \frac{1}{\sigma}\frac{1}{\sum_i n_i \frac{\langle p_i\rangle}{\langle E_i\rangle}}$$

- where we have estimated $\langle v\rangle$ as $\langle v\rangle = \langle \frac{p}{E}\rangle \simeq \frac{\langle p\rangle}{\langle E\rangle}$. Now,

$$\langle p_i\rangle = \int_0^\infty p^2 dp p \exp\left[-\sqrt{\frac{p^2+m_i^2}{T}} + \frac{\mu_i}{T}\right]$$

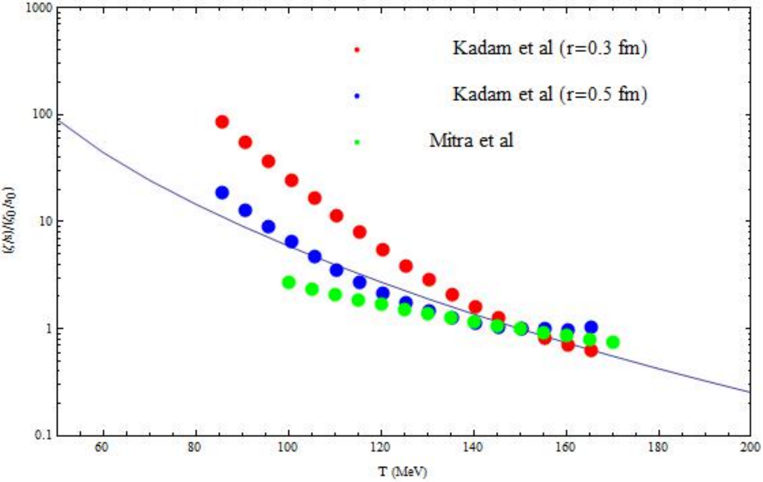
$$\langle E_i\rangle = \int_0^\infty p^2 dp \sqrt{p^2+m_i^2} \exp\left[-\sqrt{\frac{p^2+m_i^2}{T}} + \frac{\mu_i}{T}\right]$$

$$= \left(m_i^2 T^2 K_2\left(\frac{m_i}{T}\right) + \frac{m_i^3 T}{2}\left(K_1\left(\frac{m_i}{T}\right) + K_3\left(\frac{m_i}{T}\right)\right)\right) \exp\left[\frac{\mu_i}{T}\right]$$

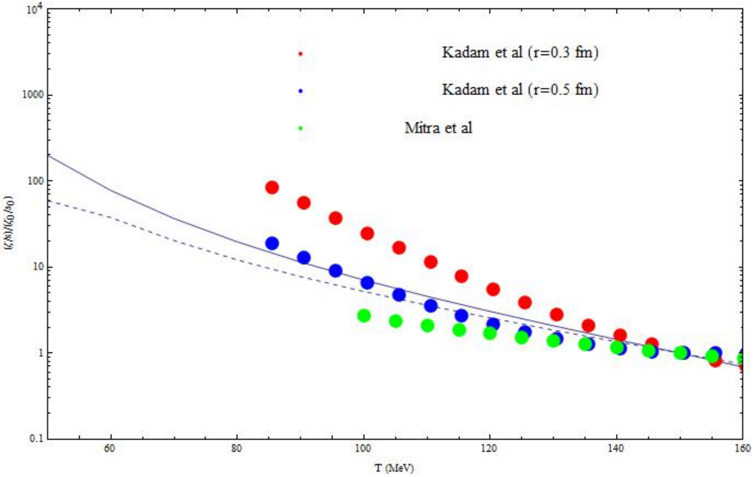
- Putting back into Eq. 2, we get

$$\tau_R^{-1} = \sigma\left(\sum_i n_i \sqrt{\frac{2m_i}{\pi T}} \left(\frac{4K_{5/2}\left(\frac{m_i}{T}\right)}{\left(\frac{m_i}{T}\right)K_1\left(\frac{m_i}{T}\right) + 2K_2\left(\frac{m_i}{T}\right) + \left(\frac{m_i}{T}\right)K_3\left(\frac{m_i}{T}\right)}\right)\right)$$

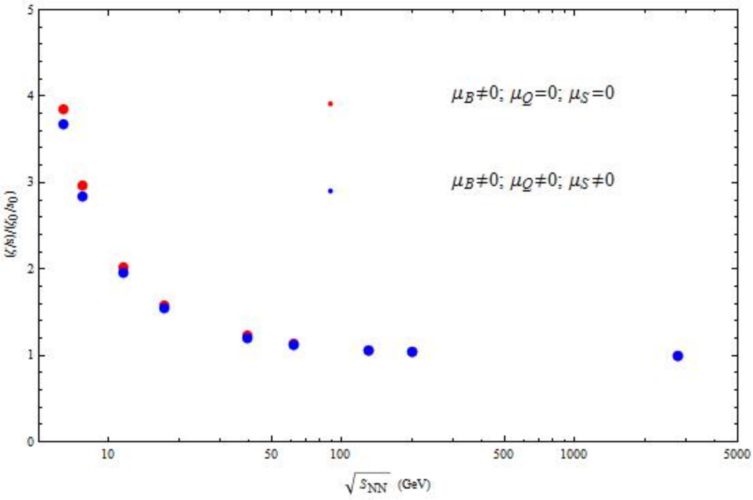
Results and Plots, Comparisons, Explanations: Bulk Viscosity with T,μ dependent τ



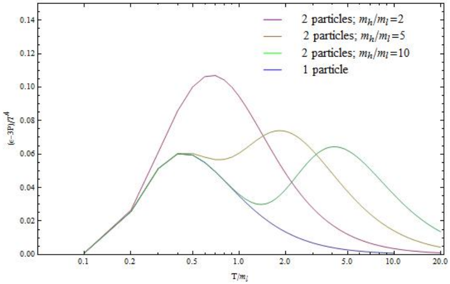
Results and Plots, Comparisons, Explanations: Bulk Viscosity with T,μ dependent τ: With Hagedrons included

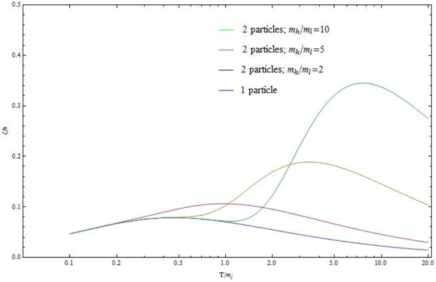


Results and Plots, Comparisons, Explanations: Bulk viscosity at different $\sqrt{s_{NN}}$

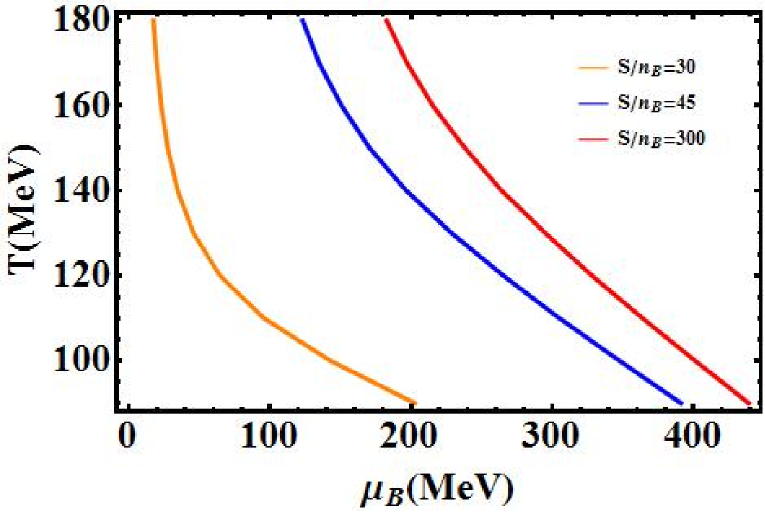


Results and Plots, Comparisons, Explanations: Understanding Bulk viscosity at different $\sqrt{s_{NN}}$ by looking at contribution to bulk viscosity from different masses:





Finding Contours in $T - \mu$ Plane in HRG phase



Summary

- **Bulk viscosity of HRG has been calculated using thermodynamic derivatives.**
- **Hagedron density of states considered.**
- **This calculation shows that at low $\sqrt{s}$ effect of Bulk viscosity viscosity is considerably large to play important role in hydrodynamic evolution of hadronic matter.**

In collaboration with:

Jan-e Alam, Sandeep Chatterjee

Thank You !